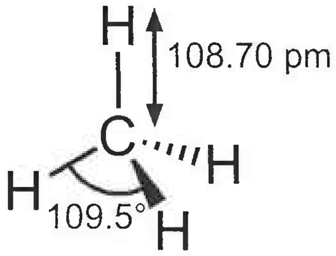
Exam Symmetry in Physics

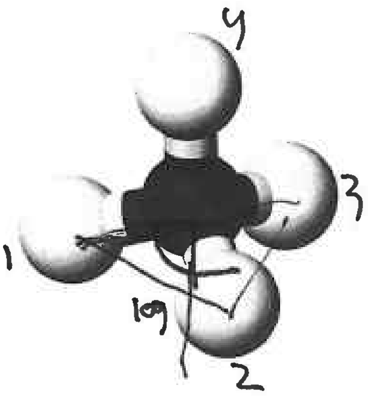
Date	June 29, 2023
Time	8:30 - 10:30
Lecturer	D. Boer

- Write your name and student number on every separate sheet of paper
- Raise your hand for more paper
- You are not allowed to use the lecture notes, nor other notes or books
- Use of a calculator is allowed
- All subquestions (a, b, c) of the three exercises have equal weight
- Illegible answers will be not be graded
- Good luck!

Exercise 1

Consider the methane molecule CH_4 depicted in the figures and its *rotational* symmetry group $G_M \cong A_4$.





(a) Identify all rotations that leave the methane molecule invariant. Reflections do not need to be considered.

(b) Divide the group elements of G_M into conjugacy classes (e.g. using geometric arguments) and determine the dimensions of the irreducible representations of G_M.

(c) Determine the characters of the three-dimensional vector representation of G_M and use them to determine whether the symmetries allow for an invariant vector, such as a permanent electric dipole moment of the molecule.

Exercise 2

Consider the group $O(3)$ of real orthogonal 3×3 matrices and its action on a rank-2 tensor σ_{ij} (i, j can take the values 1, 2, and 3). The defining (or vector) representation of $O(3)$ is the irreducible representation denoted by D^V.

(a) Show that $\sigma_{ij} = x_i y_j$ transforms according to $D^V(g)\sigma D^V(g)^T$ under $O(3)$ transformations by writing out the transformation of the vectors explicitly including the indices.

(b) Show that any $O(3)$-invariant tensor σ satisfies $[D^V(g), \sigma] = 0$ for all $g \in O(3)$ and explain that it therefore must be proportional to the Kronecker delta δ_{ij}.

(c) Determine the subgroup of $O(3)$ transformations that leave the tensor

$$\sigma_{ij} = \delta_{ij} + a\left(\delta_{i2}\delta_{j2} + \delta_{i3}\delta_{j3}\right)$$

invariant, for a nonzero constant a.

Exercise 3

Consider the Galilei transformations $T(a, b, v)$ that act in a $(1+1)$-dimensional space as follows:

$$\begin{aligned} x &\mapsto x' = x + vt + a \\ t &\mapsto t' = t + b \end{aligned}$$

for space translation a, time translation b and velocity v (all real numbers).

(a) Write $T(a, b, v)$ in the form of a three-dimensional matrix representation by letting the transformations act on vectors $(x, t, 1)$ and explain whether it is irreducible or not.

(b) Write down the inverse of $T(a, b, v)$.

(c) Explain why there is no two-dimensional matrix representation for this group in which the translations are represented non-trivially.

$$\begin{array}{ccc|ccc} & & & a & c & 0 \\ & & & b & d & 0 \\ & & & 0 & 0 & e \\ \hline a & b(1+a) & 0 & a^2+b^2(1+a) & ac+bd(1+a) & 0 \\ c & d(1+a) & 0 & ca+bd(1+a) & c^2+d^2(1+a) & 0 \\ 0 & 0 & e(1+a) & 0 & 0 & c^2(1+a) \end{array}$$

$$\begin{array}{ccc|ccc} & & & 1 & 0 & 0 \\ & & & 0 & 1+a & 0 \\ & & & 0 & 0 & 1+a \\ \hline a & b & 0 & a & b(1+a) & 0 \\ c & d & 0 & c & d(1+a) & 0 \\ 0 & 0 & e & & & \end{array}$$